Problem 2.17

A long coaxial cable (Fig. 2.26) carries a uniform *volume* charge density ρ on the inner cylinder (radius a), and a uniform *surface* charge density σ on the outer cylindrical shell (radius b). This surface charge is negative and is of just the right magnitude that the cable as a whole is electrically neutral. Find the electric field in each of the following three regions: (i) inside the inner cylinder ($s < a$), (ii) between the cylinders ($a < s < b$), (iii) outside the cable ($s > b$). Plot $|\mathbf{E}|$ as a function of s.

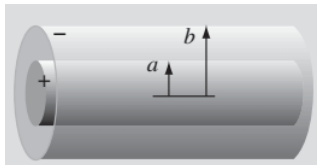


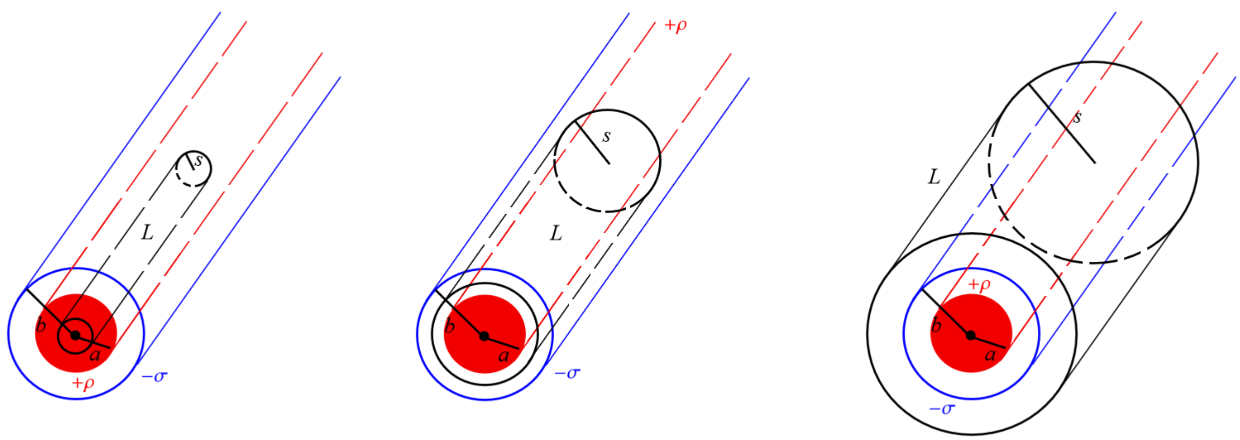
Fig. 2.26

Solution

One of the governing equations for the electric field in vacuum is Gauss's law.

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}$$

Normally the curl of $\mathbf{E}$ is also necessary to determine $\mathbf{E}$, but because of the cylindrical symmetry, the divergence is sufficient. Integrate both sides over the volume of a (black) coaxial cylindrical Gaussian surface with radius s and length L. Three cases need to be considered: one where $s < a$, one where $a < s < b$, and one where $s > b$.



$s < a$ $\quad$ $a < s < b$ $\quad$ $s > b$

The enclosed charge is the product of the charge density with the volume.

$$\int_0^L \int_0^{2\pi} \int_0^s \nabla \cdot \mathbf{E}\, dV_0 = \int_0^L \int_0^{2\pi} \int_0^s \frac{\rho}{\epsilon_0}\, dV_0 = \begin{cases} \dfrac{\rho}{\epsilon_0} \displaystyle\int_0^L \int_0^{2\pi} \int_0^s s_0\, ds_0\, d\phi_0\, dz_0 & \text{if } s < a \\ \dfrac{\rho}{\epsilon_0} \displaystyle\int_0^L \int_0^{2\pi} \int_0^a s_0\, ds_0\, d\phi_0\, dz_0 & \text{if } a < s < b \\ 0 & \text{if } s > b \end{cases}$$

The enclosed charge is zero if $s > b$ because the whole cable is electrically neutral.

Apply the divergence theorem on the left side and evaluate the integrals on the right side.

$$\int_0^L \int_0^{2\pi} (\mathbf{E} \cdot d\mathbf{S}_0)\bigg|_{s_0=s} = \begin{cases} \frac{\rho}{\epsilon_0}\left(\int_0^L dz_0\right)\left(\int_0^{2\pi} d\phi_0\right)\left(\int_0^s s_0\,ds_0\right) & \text{if } s < a \\ \frac{\rho}{\epsilon_0}\left(\int_0^L dz_0\right)\left(\int_0^{2\pi} d\phi_0\right)\left(\int_0^a s_0\,ds_0\right) & \text{if } a < s < b \\ 0 & \text{if } s > b \end{cases}$$

Because of the cylindrical symmetry, the electric field is expected to be entirely radial: $\mathbf{E} = E(s)\hat{\mathbf{s}}$. Note also that the direction of $d\mathbf{S}$ is the outward unit vector to the Gaussian surface.

$$\int_0^L \int_0^{2\pi} [E(s)\hat{\mathbf{s}}_0] \cdot (\hat{\mathbf{s}}_0\, s\, d\phi_0\, dz_0) = \begin{cases} \frac{\rho}{\epsilon_0}(L)(2\pi)\left(\frac{s^2}{2}\right) & \text{if } s < a \\ \frac{\rho}{\epsilon_0}(L)(2\pi)\left(\frac{a^2}{2}\right) & \text{if } a < s < b \\ 0 & \text{if } s > b \end{cases}$$

$$\int_0^L \int_0^{2\pi} sE(s)\,d\phi_0\,dz_0 = \begin{cases} \frac{\pi\rho s^2 L}{\epsilon_0} & \text{if } s < a \\ \frac{\pi\rho a^2 L}{\epsilon_0} & \text{if } a < s < b \\ 0 & \text{if } s > b \end{cases}$$

$$sE(s)\int_0^L \int_0^{2\pi} d\phi_0\,dz_0 = \begin{cases} \frac{\pi\rho s^2 L}{\epsilon_0} & \text{if } s < a \\ \frac{\pi\rho a^2 L}{\epsilon_0} & \text{if } a < s < b \\ 0 & \text{if } s > b \end{cases}$$

$$sE(s)\left(\int_0^L dz_0\right)\left(\int_0^{2\pi} d\phi_0\right) = \begin{cases} \frac{\pi\rho s^2 L}{\epsilon_0} & \text{if } s < a \\ \frac{\pi\rho a^2 L}{\epsilon_0} & \text{if } a < s < b \\ 0 & \text{if } s > b \end{cases}$$

$$sE(s)(L)(2\pi) = \begin{cases} \frac{\pi\rho s^2 L}{\epsilon_0} & \text{if } s < a \\ \frac{\pi\rho a^2 L}{\epsilon_0} & \text{if } a < s < b \\ 0 & \text{if } s > b \end{cases}$$

Solve for $E(s)$ by dividing both sides by $2\pi Ls$.

$$E(s) = \begin{cases} \dfrac{\rho s}{2\epsilon_0} & \text{if } s < a \\ \dfrac{\rho a^2}{2\epsilon_0 s} & \text{if } a < s < b \\ 0 & \text{if } s > b \end{cases}$$

Therefore, the electric field around the coaxial cable is

$$\mathbf{E} = \begin{cases} \dfrac{\rho s}{2\epsilon_0}\hat{\mathbf{s}} & \text{if } s < a \\ \dfrac{\rho a^2}{2\epsilon_0 s}\hat{\mathbf{s}} & \text{if } a < s < b \\ \mathbf{0} & \text{if } s > b \end{cases}.$$

Below is a plot of $E(s) = |\mathbf{E}|$ versus s.